Indian Statistical Institute, Bangalore
B. Math. Third Year, First Semester
Introduction to Stochastic Processes
Final Examination Date : Dec. 4, 2009

Maximum marks: 100 Time: 3 hours

Note: Throughout Markov chains are assumed to be time homogeneous.

1. There are two urns marked I and II. Initially, there are N red balls in urn I and N black balls in urn II, where $N \geq 2$ is a natural number. A ball is chosen at random from I and another ball is chosen at random from urn II, then they are exchanged (ball from urn I is put in urn II and vice versa). Continue this process. Let B_n be the number of black balls in urn I after n-steps. Show that $\{B_n\}_{n\geq 0}$ is a Markov chain. Compute its transition probability matrix. [15]

2. Let $\{R_n\}_{n\geq 0}$ be the symmetric random walk in one dimension, with $R_0 = 0$. What is the probability that the random walk hits 10 before it hits -2? [15]

3. Suppose customers arrive at a counter for servicing and that arrival times of customers is a Poisson process with rate λ. Suppose that service time for each customer is a fixed time $c > 0$. Initially there is no customer at the queue. Let Z_n be the number of remaining customers at the queue at the instant n-th customer got serviced and leaves the queue. Then $\{Z_n\}_{n\geq 1}$ is a Markov chain (you may assume this). Compute the transition probability matrix of this Markov chain. [20]

4. Let $\{Y_t\}_{t\geq 0}$ be a continuous time Markov process with state space $S = \{1, 2, 3\}$ and generator matrix
$$Q = \begin{bmatrix} -2 & 2 & 0 \\ 2 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
Write down the associated jump matrix and identify absorbing states, transient states and recurrent states. Compute the probability transition semigroup.
$$P_{ij}(t) = P(Y_t = j/Y_0 = i) \quad 1 \leq i, j \leq 3.$$
[20]

5. Let $\{W_t\}_{t\geq 0}$ be a poisson process with rate λ. Compute $P(W_s = m/W_t = n)$ for $0 \leq m \leq n$ and $0 < s < t < \infty$. [20]

6. Let $\{X_t\}_{t\geq 0}$ be a continuous time Markov process on a countable state space S. Show that if i leads to j, then $P_{ij}(t) > 0$ for all $t > 0$, where $P_{ij}(t) = P(X_t = j/X_0 = i)$. [20]

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